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ON THE STABILITY OF ALMOST PERIODIC SYSTEMS

Целью исследований являются почти периодические по времени в смысле Бора системы обыкновенных неавтономных дифференциальных уравнений. Предлагается решение проблем устойчивости движения на основе развития второго метода Ляпунова в классе знакопостоянных вспомогательных функций. Получены достаточные условия решения следующих задач: устойчивость, асимптотическая устойчивость (локальная и глобальная) и неустойчивость состояния равновесия. Представленные результаты теории устойчивости обобщают и дополняют известные результаты А. М. Ляпунова, Н. Г. Четаева, Е. А. Барбашина и Н. Н. Красовского для рассматриваемого типа неавтономных дифференциальных уравнений. Каждое из утверждений сопровождается иллюстрирующим примером, подтверждающим преимущества использования знакопостоянных функций по сравнению со знакоопределенными функциями Ляпунова.

Ключевые слова: почти периодическая система дифференциальных уравнений; устойчивость; функция Ляпунова.

The objects studied are almost periodic in the sense of Bohr's non-autonomous systems of ordinary differential equations. We offer a solution to the problems of stability of motion on the basis of second Lyapunov method in the class of semidefinite subsidiary functions. We give sufficient conditions for solving the following problems: stability, asymptotic stability (local and global) and instability. The presented results of the stability theory of motion generalize some known results A. M. Lyapunov, N. G. Chetaev, E. A. Barbashin and N. N. Krasovskii for this type of non-autonomous differential equations. Each of the statements is accompanied by illustrative example that confirms the advantages of the use of semidefinite functions compared with the definite functions of Lyapunov.

Key words: almost periodic system of differential equations; stability; Lyapunov function.

The method of semi definite functions as a universal way of problem-solution of the movement stability is developing for various types of dynamic processes [1–7]. The basis of this approach is the classical theorems of A. M. Lyapunov, N. G. Chetaev, E. A. Barbashin and N. N. Krasovskii, J. La Salle, V. M. Matrosov etc.

In this paper, we prove the main theorem of Lyapunov's direct method with the use of semi definite auxiliary functions for almost periodic in time (in the sense of Bohr) of ordinary differential equations. It's stated, in particular, that the almost periodic systems as a subclass of non-autonomous differential equations are correct for the theorems of the second Lyapunov's method, close to those used for the autonomous systems [1–3].

Let $\mathbb{R}^n$ be a n-dimensional Euclidean space, $\|\cdot\|$ is a norm in this space, $\mathbb{R}^+$ is a set of nonnegative real numbers of a real straight line $\mathbb{R}$; $\mathbb{N}$ is a set of natural numbers; $B_r = \{x \in \mathbb{R}^n : \|x\| < r\}$ is a sphere with the radius of $r > 0$.

Consider a system of non-autonomous differential equations

$$\dot{x} = f(x,t)\,,\ x \in D \in \mathbb{R}^n\,,\ t \in \mathbb{R}\,, \tag{1}$$

defined in an connected neighborhood D of the origin. Suppose that the function f: $D \times \mathbb{R} \to \mathbb{R}^n$ is continuous, satisfies the local Lipschitz condition and $f(0, t) = 0\ \forall\ t \in \mathbb{R}$.

Whereas it is known that through each point (x_0, t_0) of domain of (1) passes a unique solution $x(x_0, t_0, t)$, with initial condition $x(x_0, t_0, t_0) = x_0$. For each solution $x(x_0, t_0, t)$ the symbols $\gamma^{\pm}(x_0, t_0) = \{y \in \mathbb{R}^n : y = x(x_0, t_0, t),\ t \in \mathbb{R}^{\pm}\}$ denote respectively the positive and negative semi-trajectory.

Almost periodic systems. Let us first recall the following definitions [8, 9]. Number set $\Xi = \{\xi\}$ is called relatively dense on $\mathbb{R}$ if there exists $l > 0$ where every segment $a \le x \le a + l$ of length l contains at least one element of the set Ξ, i. e., for any a we have $[a, a+l] \cap \Xi \ne \varnothing$.

Number $\tau = \tau(\varepsilon)$ is called *an almost period* of continuous function $g : \mathbb{R} \to \mathbb{R}^n$ with up to ε (in short: it ε-*almost period*) if $\left|g(t+\tau) - g(t)\right| < \varepsilon\ \ \forall\ t \in \mathbb{R}$. Continuous function $g : \mathbb{R} \to \mathbb{R}^n$ is called *almost periodic* (in the sense of Bohr) if for every $\varepsilon > 0$, there is a relatively dense set of almost periods τ for function $g(t)$ with up to ε. Continuous function $f(x,t)$ ($f : \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^n$) is called *uniformly almost periodic*, if every $\varepsilon > 0$ and every $r > 0$ corresponds such $L = L(\varepsilon, r) > 0$ that in any interval $[a, a + L(\varepsilon, r)]$, $a \in \mathbb{R}$, there is at least one number τ, where $\| f(x, t+\tau) - f(x,t) \| < \varepsilon$ if $t \in \mathbb{R}$, $\|x\| < r$. It's well known [9] that if the function $f(x,t)$ is almost periodic in t and satisfies the local Lipschitz condition in x, then it is uniformly almost periodic. Further it will be assumed that the function $f(x,t)$ is (uniformly) almost periodic in time t. Let us recall the following results.

Statement 1 [9]. *Let the functions* $f : D \times \mathbb{R} \to \mathbb{R}^n$ *and* $V : D \times \mathbb{R} \to \mathbb{R}$ *are continuous and almost periodic in time t. Then for any* $\varepsilon > 0$ *and* $r > 0$, *there exists a sequence of* ε-*almost periods* (τ_k) $(\tau_k \to +\infty)$ *common for functions* $f(x,t)$ *and* $V(x,t)$ *such that for all* $(x, t) \in B_r \times \mathbb{R}$ *we have inequalities*

$$\| f(x,t) - f(x, t+\tau_k) \| < \varepsilon, \quad | V(x,t) - V(x, t+\tau_k) | < \varepsilon. \tag{2}$$

Let us show the modification of one of the results [9].

Statement 2. *Let the solution* $x(x_0, t_0, t)$ *of* (1) *is located in the sphere* B_r $(\overline{B_r} \subset D)$, $r > 0$, *at all* $t \in]t_0 - a, +\infty[$, *where* $a > 0$ (ε_k) *is monotonically vanishing sequence of positive numbers*, (τ_k) *is a sequence of* ε_k-*almost periods of the function* $f : \underline{D} \times \mathbb{R} \to \mathbb{R}^n$ (*each* ε_k *corresponds* ε_k-*almost period* τ_k). *Then, for any fixed time* $t^* > t_0 - a$ *uniformly in* $x_0 \in B_r$ *the limit relation is done*

$$\lim_{k\to\infty}\| x(x_k,t_0,t^*)-x(x_0,t_0,t^*+\tau_k)\|=0, \tag{3}$$

where $x_k = x(x_0,t_0,t_0+\tau_k)$. *Or*, *equivalently*,

$$\lim_{k\to\infty}\| x(x(x_0,t_0,t_0+\tau_k),t_0,t^*)-x(x(x_0,t_0,t_0+\tau_k),t_0+\tau_k,t^*+\tau_k)\|=0\,. \tag{4}$$

In the work [10] the following result is proved.

Lemma 1. *Let* $f(x,t)$ *is a uniform almost periodic function of time t*, (ε_k) *is monotonically vanishing sequence of positive numbers*, (τ_k) *is a sequence of* ε_k*-almost periods of* $f:D\times\mathbb{R}\to\mathbb{R}^n$ (*for every* ε_k *there is* ε_k*-almost period* τ_k *and* $\tau_k\to+\infty$ *if* $k\to+\infty$). *Then*, *if the solution* $x = 0$ *of* (*1*) *is unstable*, *there are numbers* $\varepsilon > 0$ *and* $0 < \delta < \varepsilon$, *and there exists a sequence of states* (x_{0n}) ($x_{0n}\to 0$) *such that:*

1) $\| x(x_{0n},0,t)\|<\delta$ *for* $0\le t<\tau_n$ *and* $\delta\le\| x(x_{0n},0,t)\|<\varepsilon \quad \forall n\in\mathbb{N}$;

2) $\gamma^-(y^0,0)\subset\overline{B_\varepsilon}$ *for* $y^0=\lim\limits_{n\to+\infty}x(x_{0n},0,\tau_n)$ ($\delta\le\| y^0\|\le\varepsilon$).

The direct method of Lyapunov. Let us recall the definitions of a stability theory [2, 5, 7]. The solution $x = 0$ of the equation (1) is

– *stable* if $(\forall t_0 \ge 0)(\forall\varepsilon > 0)(\exists\delta = \delta(t_0, \varepsilon) > 0)(\forall x_0 \in B_\delta) \Rightarrow \| x(x_0,t_0,t)\| < \varepsilon\ \forall t \ge t_0$;

– *attractive* if $(\forall t_0 \ge 0)(\exists\sigma = \sigma(t_0) > 0)(\forall\alpha > 0)\ (\forall x_0 \in B_\sigma)(\exists T(t_0, \alpha, x_0) > 0) \Rightarrow \| x(x_0,t_0,t)\| < \alpha\ \forall t \ge t_0 + T$;

– *asymptotically stable* if it is stable and attractive;

– *globally asymptotically stable* if it is asymptotically stable, $D = \mathbb{R}^n$, and each solution $x(x_0,t_0,t)$, $x_0 \in \mathbb{R}^n$, $t_0 \ge 0$, of the equation (1) at $t \to +\infty$ tends to the origin of coordinates;

– *unstable* if $(\exists t_0 \ge 0)(\exists\varepsilon > 0)(\forall\delta > 0)(\exists x_0 \in B_\delta)(\exists t^* > t_0) \Rightarrow \| x(x_0,t_0,t^*)\| \ge \varepsilon$.

Now let us state and prove the main theorems of Lyapunov's second method in the class of semi definite functions.

Theorem 1. *Suppose that for almost periodic system* (*1*) *there exist a neighborhood U of* $x = 0$, *continuously differentiable*, *almost periodic in time t the function* $V:U\times\mathbb{R}\to\mathbb{R}^+$ *such that*, *for all* $(x,t)\in U\times\mathbb{R}$ *the following conditions are done*:

1) $V(x,t)\ge 0$ *and* $V(0,t)=0$;

2) $\dot V(x,t)\le 0$;

3) *the set* $Y_0=\overline{\{x\in U:\ V(x,t)=0\ \forall t\in\mathbb{R}\}}$ *does not contain negative semi-trajectories of* (*1*), *except at* $x = 0$. *Then zero solution of* (*1*) *is stable*.

Proof. Let the assumptions of the Theorem 1 are performed and let, on the contrary, zero solution of equation (1) is unstable. Then by Lemma 1 there exist $\varepsilon > 0$ and $0 < \delta < \varepsilon$, there is a sequence of states (x_{0n}), $x_{0n}\to 0$, where the conditions 1) and 2) of Lemma 1 are performed, that is, non-zero semi-trajectory $\gamma^-(y^0,0)\subset\overline{B_\varepsilon}$. We will show that $\gamma^-(y^0,0)\subset Y_0$. To do this, without loss of generality, we assume that $\overline{B_\varepsilon}\subset U$. Fix time $t < 0$. Then, since $\tau_n\to+\infty$, then for all sufficiently large n ($n \ge N$) it will be $\tau_n + t > 0$. Moreover, by conditions 1) and 2) of Theorem 1, we can write the relations

$$0\le V(x(x(x_{0n},0,\tau_n),\tau_n,\tau_n+t),\tau_n+t)\le V(x_{0n},0)\quad \forall n\ge N\,. \tag{5}$$

By the Statement 1 the condition (2) takes place and, hence, from (5) we obtain

$$0\le V(x(x(x_{0n},0,\tau_n),\tau_n,\tau_n+t),t)+\eta_n\le V(x_{0n},0)\quad n\ge N\,.$$

where $\eta_n\to 0$ at $n\to+\infty$.

Note that, by the Statement 2, and the property of continuous dependence of solutions at the initial data, the points $x(x(x_{0n},0,\tau_n),\tau_n,\tau_n+t)$ and $x(x(x_{0n},0,\tau_n),0,t)$ will be close at $n\to+\infty$. Therefore, it follows from the above that

$$0\le V(x(x(x_{0n},0,\tau_n),0,t),t)+\mu_n+\eta_n\le V(x_{0n},0)\quad n\ge N\,,$$

where $\mu_n\to 0$ at $n\to+\infty$. Hence, passing to the limit $n\to+\infty$, we'll get the identity

$$V(x(y^0,0),t)=0\ \ \forall t<0\,.$$

According to a choice of the moment $t < 0$ the last means that $\gamma^-(y^0,0)\subset Y_0$, $y^0\ne 0$. However, this contradicts to 3) that proves the Theorem 1.

Theorem 2. *Suppose that for almost periodic system* (*1*) *there exist a neighborhood U of* $x = 0$, *continuously differentiable*, *almost periodic in time t function* $V:U\times\mathbb{R}\to\mathbb{R}^+$ *such that*, *for all* $(x,t)\in U\times\mathbb{R}$ *the following conditions are performed*:

1) $V(x,t) \geq 0$ *and* $V(0,t) = 0$;

2) $\dot{V}(x,t) \leq 0$;

3) the set $Y = \overline{\{x \in U : \dot{V}(x,t) = 0 \ \forall t \in \mathbb{R}\}}$ *does not contain negative semi-trajectories of (1), except at* $x = 0$. *Then zero solution of (1) is asymptotically stable.*

Proof. Since, by definition, the set Y_0 of the Theorem 1 is contained in the set Y of the Theorem 2, the stability of the zero solution of (1) follows from the Theorem 1. Let quantities $\varepsilon > 0$, $t_0 \in \mathbb{R}$, and $\delta = \delta(\varepsilon, t_0) > 0$ satisfy the definition of stability in the sense of Lyapunov, and we assume that $\overline{B_\varepsilon} \subset U$.

We'll show that any solution $x(x_0, t_0, t)$ with the initial state $\| x_0 \| < \delta$, tends to the origin. Indeed, if this is not done, so with respect to the stability property this solution is strictly separated from zero at $t > t_0$, that is, there exists such a number $\mu > 0$ that we have the inequality

$$0 < \mu \leq \| x(x_0, t_0, t) \| < \varepsilon \quad \forall t \geq t_0. \tag{6}$$

Let (ε_k), $\varepsilon_k \to 0$, is a sequence of positive numbers and $(\tau_{k,m}) \subset \mathbb{R}^+$ – corresponding sequence of almost periods such that $\tau_{k,m} \to +\infty$ at $m \to +\infty$ and for functions $f(x,t)$ and $V(x,t)$ the relations (2) are performed. Without a loss of generality we consider that $\tau_{k,m} \leq \tau_{k+1,m}$ for all k, m and assume that $\tau_k = \tau_{k,k}$. Consider the sequence of points $(x_k)_{k \geq 1}$, defined by the formula

$$x_k = x(x_0, t_0, t_0 + \tau_k). \tag{7}$$

On the basis of (6), the sequence (x_k) is bounded, and so we can consider that it converges, i. e.,

$$x^* = \lim_{k \to \infty} x(x_{0k}, t_0, t_0 + \tau_k). \tag{8}$$

It is obvious that $x^* \in \overline{B_\varepsilon}$.

As $V(x,t)$ is continuous and almost periodical we have:

$$V\left(x^*, t_0\right) = \lim_{n \to +\infty} V\left(x_n, t_0\right) = \lim_{k \to +\infty} \lim_{n \to +\infty} V\left(x_n, t_0 + \tau_k\right) = \lim_{n \to +\infty} V\left(x_n, t_0 + \tau_n\right) =$$
$$= \lim_{n \to +\infty} V\left(x\left(x_0, t_0, t_0 + \tau_n\right), t_0 + \tau_n\right) = V_0. \tag{9}$$

Consider the solution $x(x^*, t_0, t)$ and show that it is entirely located if the $t < t_0$ on the set Y. In fact, if it is not so, then, according to the condition 2) and (9) there is a time instant $t^* < t_0$, for which we have the inequality $V\left(x\left(x^*, t_0, t^*\right), t^*\right) = V_1 > V_0$.

On the other hand, due to the limit (8) and continuous dependence on the initial conditions, the following equation is right:

$$x(x^*, t_0, t^*) = \lim_{k \to \infty} x(x_k, t_0, t^*),$$

and, hence, taking into account the continuity of $V(x(x_k, t_0, t), t)$ it follows that

$$\lim_{k \to \infty} V\left(x\left(x_k, t_0, t^*\right), t^*\right) = V_1. \tag{10}$$

Further, referring to the properties of almost periodic system (1) from the limit relations (3) and (4) we obtain the condition

$$\| x(x_k, t_0, t^*) - x(x_0, t_0, t^* + \tau_k) \| < \nu_k, \tag{11}$$

where $\nu_k \to 0$. In addition, according to the almost periodicity of $V(x,t)$ it follows that

$$| V(x, t^*) - V(x, t^* + \tau_k) | < \varepsilon_k, \tag{12}$$

and therefore from (10) and (11) we obtain

$$\left| V\left(x\left(x_0, t_0, t^* + \tau_k\right), t^*\right) - V_1 \right| < \eta_k, \ \eta_k \to 0. \tag{13}$$

On the other hand, on the base of (12) we have

$$\left| V\left(x\left(x_0, t_0, t^* + \tau_k\right), t^*\right) - V\left(x\left(x_0, t_0, t^* + \tau_k\right), t^* + \tau_k\right) \right| < \varepsilon_k. \tag{14}$$

Both conditions (13) and (14) give the inequality

$$| V(x(x_0, t_0, t^* + \tau_k), t^* + \tau_k)) - V_1 | < \eta_k + \varepsilon_k, \tag{15}$$

where $\eta_k + \varepsilon_k \to 0$ at $k \to \infty$.

Thus, by construction

$$\lim_{k \to \infty} V\left(x\left(x_k, t_0, t^* + \tau_k\right), t^* + \tau_k\right) = V_0. \tag{16}$$

However, the simultaneous fulfillment of (15) and (16) is impossible on the base of given above inequality $V_1 > V_0$. Thus, the semi-trajectory $\gamma^-(x^*, t_0)$ should be placed on a set of Y, where $\| x^* \| \geq \mu > 0$. This statement contradicts the assumption 3) of Theorem 1. And this contradiction completes the proof.

Theorem 3. *Suppose that* $D=\mathbb{R}^n$ *and for almost periodic system (1) there exist a continuously differentiable, almost periodic in time t function* $V:\mathbb{R}^n\times\mathbb{R}\to\mathbb{R}^+$ *where for all* $(x,t)\in\mathbb{R}^n\times\mathbb{R}$ *the following conditions are performed*:

1) $V(x,t)\geq 0$ *and* $V(0,t)=0$;

2) $\dot V(x,t)\leq 0$;

3) The set $Y=\overline{\{x\in\mathbb{R}^n:\ \dot V(x,t)=0\ \forall t\in\mathbb{R}\}}$ *does not contain relatively compact negative semi-trajectories of* (*1*), *except at* $x=0$.

Then zero solution of (*1*) *is globally asymptotically stable*.

Proof. Since all the conditions of the Theorem 2 are done, zero solution of (1) is asymptotically stable. Now it's necessary to show that for every pair of initial conditions $(x_0,t_0)\in\mathbb{R}^n\times\mathbb{R}$ the solution $x(x_0,t_0,t)$ tends to the origin if $t\to+\infty$.

Indeed, if this property is not performed, with the asymptotic stability of the origin and condition 4) of Theorem 3 we can write the relation, as follows

$$0<\eta\leq\|x(x_0,t_0,t)\|\leq M<+\infty,\ \forall t\geq t_0. \tag{17}$$

Applying arguments similar to those used in the proof of the Theorem 2, in the same way we can obtain a contradiction with the assumption (17). It also completes the proof.

Theorem 4. *Suppose that for almost periodic system* (*1*) *there exist a neighborhood U of* $x=0$, *continuously differentiable*, *almost periodic in time t function* $V:U\times\mathbb{R}\to\mathbb{R}$ *where for all* $(x,t)\in U\times\mathbb{R}$ *the following conditions are hold*:

1) $V(x,t)\geq 0$ *and* $V(0,t)=0$;

2) $\forall\alpha>0\ \exists\tau\in\mathbb{R}$ *and* $\exists p\in B_\alpha$ *such that* $V(p,\tau)>0$;

3) $\dot V(x,t)\geq 0$;

4) $x(x_0,t_0,t)\to 0$ *at* $t\to+\infty$ *if the solution* $x(x_0,t_0,t)\in U$, $\dot V(x(x_0,t_0,t),t)=0$ *and* $V(x(x_0,t_0,t),t)>0$ *for all* $t\geq t_0$.

Then zero solution of (*1*) *is unstable.*

Proof. We'll fix ε > 0 so that $\overline{B_\varepsilon}\subset U$. Let δ > 0 is arbitrarily small number less than ε. Due to condition 2) there exists a pair $(x_0,t_0)\in B_\delta\times\mathbb{R}$ where $V(x_0,t_0)>0$. We'll show that the solution $x(x_0,t_0,t)$ leave the sphere B_ε at some $t>t_0$ and it will confirm the validity of the theorem.

Indeed, suppose that, on the contrary, the opposite property is performed, i.e., $\|x(x_0,t_0,t)\|<\varepsilon\ \forall t\geq t_0$.

From 1) and continuity and boundedness in t $V(x,t)$ follows the existence of η > 0 such that $|V(x,t)|<V(x_0,t_0)\ \ \forall t\geq t_0$, if $\|x\|<\eta$. In this case by condition 3) we have the inequality $\|x(x_0,t_0,t)\|\geq\eta$ $\forall t\geq t_0$. Thus, the considered solution $x(x_0,t_0,t)$ is subjected to the following condition:

$$0<\eta\leq\|x(x_0,t_0,t)\|<\varepsilon\ \ \forall t\geq t_0. \tag{18}$$

Replicating with a negligible change in the proof of the Theorem 2 (from (5)), but in this case at time $t^*>t_0$, we can see that the solution $x(x^*,t_0,t)$, where the state x^* is defined by (6) and (7), located on the set where $\dot V=0$ and $V>0$ at all $t\geq t_0$. Hence, on the basis of the requirement 4) of Theorem 4 it follows that $x(x^*,t_0,t)\to 0$ at $t\to+\infty$. But in this case (18) and (6) are contradictory. And this contradiction proves Theorem 4.

Example. Consider in the space $\mathbb{R}^2$ the system of differential equations

$$\dot x=3y^3,\quad \dot y=-b(t)(x+y^3)(y+a(x,y,t))^2-y, \tag{19}$$

where $a(x,y,t)$ and $b(t)$ are continuous, almost periodic in *t*. We'll take the semi definite function $V(x,y)=$ $=(x+y^3)^2$ with the time derivative equaled to

$$\dot V(x,y)=-3b(t)y^2(x+y^3)^2(x+a(y,t))^2.$$

We'll require that the inequality is

$$b(t)\geq 0\quad\forall t\in\mathbb{R}, \tag{20}$$

and we'll show that a zero solution of (19) is stable. Indeed, the set Y_0, where $V=0$, performed by $Y_0=\{(x,y)\in\mathbb{R}^2:$ $x=-y^3\}$. On this set the system is described by a scalar equation

$$\dot y=-y, \tag{21}$$

for which every nonzero negative semi-trajectory leaves every fixed neighborhood of the origin. Therefore, according to the assumptions (20), all conditions of Theorem 1 will be done, where the origin of the system (19) is stable.

We'll give the conditions for which the original system is *asymptotically stable*. For this we require this strict inequality

$$b(t) > 0 \quad \forall t \in \mathbb{R}. \tag{22}$$

Note that according to (22) the set Y, where $\dot{V} = 0$, is given by

$$Y = Y_0 \cup \{(x, y) \in \mathbb{R}^2 : x = -a(y, t),\ b(t) > 0,\ t \in \mathbb{R}\} \cup \{(x, y) \in \mathbb{R}^2 : y = 0\}.$$

Therefore, to verify the condition 3) of Theorem 2 it is sufficient to consider two subsets: $x = -a(y, t)$ and $y = 0$. We can see that on the first of these subsets of the original system of equations goes into scalar differential equation (21). Hence, this set does not contain negative semi-trajectories, except zero.

In the case of $y = 0$, the system is defined by

$$\dot{x} = 0, \quad 0 = -b(t)a^2(0,0,t)x. \tag{23}$$

Let the following condition takes place

$$a(0,0,t) \neq 0 \quad \forall t \in \mathbb{R}. \tag{24}$$

Then from (23) follows that $x = 0$. In other words, under condition (24), the set $y = 0$ contains only zero trajectories, and hence in this case condition 3) of Theorem 2 will also be performed. In the result we get that conditions (22) and (24) guarantee the asymptotic stability of the solution $x = 0$, $y = 0$ of the system (19).

Now we shall determine the assumptions under which the zero solution of (19) with conditions (22), (24) is *globally asymptotically stable*. To do this we'll show that every solution $(x(t), y(t))$ is bounded if the function a satisfies the following conditions:

$$\forall A > 0, \exists B > 0,\ |y| > A \Rightarrow |y + a(x, y, t)| < B \quad \forall x \in \mathbb{R}, \forall t \in \mathbb{R}. \tag{25}$$

Indeed, the function V does not increase along the solutions and therefore the expression $|x(t) + y^3(t)|$ is bounded for all t of $\mathbb{R}^+$. So as almost periodic function $b(t)$ is always bounded, then we have

$$y(t)\dot{y}(t) = -b(t)y(t)(x(t) + y^3(t))\big(y(t) + a(x, y, t)\big)^2 - y^2(t) \leq 0$$

for sufficiently large values $|y(t)|$. This proves the boundedness of the component $y(t)$ at $t > 0$. But then, from the boundedness of $|x(t) + y^3(t)|$ and $|y(t)|$ the boundedness of the solution $(x(t), y(t))$ if $t > 0$ and in the component $x(t)$ follows. Thus, the conditions (22), (24) and (25) ensure all the requirements of Theorem 3, and hence, zero solution of the original system is globally asymptotically stable.

Finally, we'll define the *conditions of instability*. Let the inequality

$$b(t) < 0 \quad \forall t \in \mathbb{R} \tag{26}$$

takes place.

Then the above mentioned Lyapunov's function V together with its derivative with respect to time $\dot{V}$ will satisfy the hypothesis 1), 2) and 3) of the Theorem 4.

To perform condition 4) of Theorem 4, we'll note the following. On Y system (19) is described either by differential equation (21) or by the differential equation (23). As for the equation (21), all solutions tend asymptotically to the origin of coordinates. Therefore, if a certain solution $(x(t), y(t))$ of the system satisfies (21), it does not contradict to hypothesis 4) of Theorem 4. If the solution $(x(t), y(t))$ of the system satisfies the equation (23) and the condition (24) is performed, so the solution can only be trivial, i. e., $x = 0$, $y = 0$.

Thus, according to the hypotheses (24) and (26) the condition 4) of Theorem 4 will be satisfied, and hence zero solution of (19) is unstable.

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